Conformal Enhancement of Holographic Scaling in Black Hole Thermodynamics: A Near-Horizon Heat-Kernel Framework

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ABSTRACT: Standard thermodynamic treatments of quantum field theory in the presence of black-hole backgrounds reproduce the black hole entropy by usually specializing to the leading order of the heat-kernel or the high-temperature expansion. By contrast, this work develops a hybrid framework centered on geometric spectral asymptotics whereby these assumptions are shown to be unwarranted insofar as black hole thermodynamics is concerned. The approach—consisting of the concurrent use of near-horizon and heat-kernel asymptotic expansions—leads to a proof of the *holographic scaling of the entropy* as a universal feature driven by conformal quantum mechanics.

KEYWORDS: Black Holes, Models of Quantum Gravity.

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1. Introduction: thermodynamics and heat kernel approach

The scaling of the black hole entropy with the area of the event horizon has been confirmed by multiple lines of research [1, 2]. Its existence and robustness suggest that the relevant gravitational states come in a denumerable set associated with a Planck-area partitioned horizon [2, 3, 4]. This is a holographic property: the horizon encodes information at the quantum level—with generalizations to a holographic principle [5] and the AdS/CFT correspondence [6]. Conversely, the near-horizon physics of black holes may provide guiding hints into the nature of quantum gravity. As a first "semiclassical" approach, the quantum fields act as probes of the gravitational background, and an *ultraviolet catastrophe* [2] affects the spectral and thermodynamic functions. In its most basic format, the mandatory regularization is called the brick-wall model [3], with the dominant part of the entropy arising from within a Planck-length skin of the horizon—a method equivalent to the entanglement entropy [7]. Most importantly, a universal near-horizon behavior, known as *conformal quantum mechanics* (CQM), appears to drive the thermodynamics [8, 9].

In this paper we explore the near-horizon emergence of black hole thermodynamics with the aid of the *heat-kernel method*. Specifically, our framework is based on a direct evaluation of the spectral functions to highlight the ultraviolet catastrophe and the concomitant role of CQM. As we show below, the near-horizon CQM contribution to black hole thermodynamics trades the orthodox hierarchy in favor of a *universal holographic scaling* that weighs similarly all terms in the heat-kernel expansion, with the entropy being proportional to the horizon area. This *conformal enhancement* reveals the holographic nature of the entropy and entails a revision of semiclassical treatments of quantum field theory near an event horizon.

The proposed framework is motivated by the need to go beyond the "thermodynamic limit" of regular systems, in which boundary contributions are neglected compared to the bulk. In effect, as boundaries become prominent, e.g., for the Casimir effect [10] or for

mesoscopic scales, boundary corrections are also required. In addition, in the presence of a nontrivial bulk geometry, curvature-dependent corrections set in. For black-hole backgrounds, one would expect both types of corrections to be relevant *a priori*. The ensuing *geometric spectral asymptotics* involves a hierarchy of curvature and boundary contributions organized by the heat kernel formalism [10, 11, 12], which starts with the equation

$$\mathcal{K}\,\Phi = \lambda\,\Phi\,, \tag{1.1}$$

from which the operator $\mathcal{K}$ yields the system's spectral functions. Then, the heat-kernel trace, $Y_{\mathcal{K}}(\tau) \equiv \mathrm{Tr}\,\exp\left(-\tau\mathcal{K}\right)$, provides the main asymptotic expansion,

$$Y_{\mathcal{K}}(\tau) \overset{(\tau\to 0)}{\sim} \sum_{j=0}^{\infty} a_j(\mathcal{K})\;\tau^{r_j(d)}\;, \tag{1.2}$$

where $r_j(d) = (j-d)/2$ and d is the relevant dimensionality [10, 11, 12]—in our canonical framework described below, after a dimensional reduction, d will be the spatial dimension.

Our goal is to build the thermodynamics *ab initio* and bypass the assumptions built into the standard thermodynamic treatments of quantum field theory in black-hole backgrounds. In the proposed framework, the density of modes $\rho_{\mathcal{K}}(\lambda)$ is a primary spectral function for the system's statistical mechanics. As $Y_{\mathcal{K}}(\tau) = \int_0^\infty d\lambda\,\exp\left(-\tau\lambda\right)\,\rho_{\mathcal{K}}(\lambda)$ [for a spectrum in $[0,\infty)$], the asymptotic expansion of $\rho_{\mathcal{K}}(\lambda)$ is the formal inverse Laplace transform of eq. (1.2); then, the *spectral counting function* measuring the cumulative number of modes is

$$N_{\mathcal{K}}(\lambda) \equiv \int^{\lambda} d\lambda' \rho_{\mathcal{K}}(\lambda') \overset{(\lambda\to\infty)}{\sim} \sum_{j=0}^{\infty} \frac{a_j(\mathcal{K})}{\Gamma(\,-\,r_j(d)+1)}\,\lambda^{-r_j(d)}\;. \tag{1.3}$$

In particular, with the relevant metric $\tilde{\gamma}_{ij}$ [to be defined in conjunction with eq. (1.1)],

$$a_0 \equiv a_0(\mathcal{K}) = \frac{1}{(4\pi)^{d/2}} \int_{\mathcal{M}} d^d x\,\sqrt{\tilde{\gamma}} \tag{1.4}$$

yields Weyl's asymptotic formula [10]

$$N^{(0)}_{\mathcal{K}}(\lambda) \overset{(\lambda\to\infty)}{\sim} \frac{\tilde{\mathcal{V}}_d\left(\mathcal{M}\right)}{(4\pi)^{d/2}\,\Gamma\left(1+d/2\right)}\,\lambda^{d/2} = (2\pi)^{-d}\;\mathcal{B}_d\,\tilde{\mathcal{V}}_d\left(\mathcal{M}\right)\;\lambda^{d/2}\;, \tag{1.5}$$

in terms of the manifold volume $\tilde{\mathcal{V}}_d\left(\mathcal{M}\right)$ and the volume $\mathcal{B}_d$ of the unit ball. Equation (1.5) reproduces Euclidean-space bulk thermodynamics and agrees with the mode-counting algorithms of standard WKB, phase-space methods, and the brick-wall model. In this paper: (i) we generalize the latter by going beyond eq. (1.5) via the inclusion of terms with $j \neq 0$; and (ii) show that the orthodox heat-kernel hierarchy breaks down in the presence of an event horizon, leading to the emergence of universal holographic scaling. In a sense, this is reminiscent of the metaphor "hearing the shape of a drum" [13]: for black holes, when one is effectively *"probing the event horizon,"* the output is invariably a holographic response.

2. Quantum field theory: canonical reduction procedure

For our current purposes, we take the broad family of generalized Schwarzschild metrics,

$$ds^2 = -f(r)\, dt^2 + [f(r)]^{-1}\, dr^2 + r^2\, d\Omega^2_{(D-2)} \ , \tag{2.1}$$

in D spacetime dimensions, as a background probe for quantum fields. For the sake of simplicity, we consider one species of a scalar field, with action ($D \geq 4$)

$$S = -\frac{1}{2} \int d^D x \sqrt{-g}\, \left[g^{\mu\nu}\, \nabla_\mu \Phi\, \nabla_\nu \Phi + m^2 \Phi^2 + \xi R \Phi^2 \right] \ , \tag{2.2}$$

and generalizations that will be discussed elsewhere. We begin our first-principle framework with a multiple reduction procedure that consists of the following three steps:

1. The reduction of the original D-dimensional spacetime geometry to a d-dimensional spatial (with $d = D - 1$) geometry, along with a Fourier resolution of the fields.

2. A conformal transformation of the metric (2.1),

$$\tilde{g}_{\mu\nu} = \Omega^2\, g_{\mu\nu} \ , \tag{2.3}$$

with the conformal factor $\Omega^2 = 1/|g_{00}| = 1/f(r)$.

3. A near-horizon approximation with respect to the shifted radial coordinate

$$x = r - r_+ \tag{2.4}$$

away from the horizon $\mathcal{H}$ ($r = r_+$), with the largest root r_+ of $g^{rr}(r) = f(r) = 0$.

In the first step of the reduction procedure, a dimensional reduction trades covariance in favor of a *constructive thermodynamic approach* based on frequency-mode counting. The static metric (2.1) yields the Killing vector ∂_t, orthogonal to spacelike hypersurfaces that foliate spacetime, and permits the separation of the coordinate t via the basis of modes $\phi_{s,\pm\omega}(t, \vec{x}) = \phi_s(\vec{x}) e^{\mp i\omega t}$ (with $\omega \equiv \omega_s$) satisfying the Lie-derivative equation $\mathcal{L}_{\partial_t}\, \phi_{s,\pm\omega}(t, \vec{x}) = \mp i\, \omega\, \phi_{s,\pm\omega}(t, \vec{x})$ and the Klein-Gordon equation $\left[\Box - \left(m^2 + \xi R \right) \right] \phi_{s,\pm\omega} = 0$. Then, in the generalized-Schwarzschild frame, the quantum-field-operator expansion becomes [8, 9]

$$\Phi(t, \vec{x}) = \sum_s \left[a_s\, \phi_s(\vec{x})\, e^{-i\omega t} + a_s^\dagger\, \phi_s^*(\vec{x})\, e^{i\omega t} \right] \ , \tag{2.5}$$

while the Klein-Gordon equation takes the form

$$-\Delta_{(\gamma)} \phi - 2\, \gamma^{ij} \omega_i \partial_j \phi + \left(m^2 + \xi R \right) \phi = \frac{\omega^2}{f(r)}\, \phi \ , \tag{2.6}$$

where $\Delta_{(\gamma)}$ is the Laplace-Beltrami operator in the spatial metric $\gamma_{ij} dx^i dx^j = [f(r)]^{-1}\, dr^2 + r^2\, d\Omega^2_{(D-2)}$ and $\omega_i = \frac{1}{2}\, \partial_i \left(\ln \sqrt{|g_{00}|} \right)$ is a one-form (from dimensional reduction).

The second step of the reduction procedure originates in two complicating features of the Klein-Gordon equation (2.6): (i) first-order derivatives; (ii) the placement of the

function $f(r)$ on its right-hand side. The latter feature suggests the need for a transformation that recasts eq. (2.6) into the form (1.1), with the eigenvalue λ as function of the frequency ω (for the thermodynamic counting of modes). The resulting conformal transformation (2.3) of eq. (2.1) defines the spacetime *optical metric* [14, 15, 16], whose spatial part $\tilde{\gamma}_{ij} = [f(r)]^{-1} \gamma_{ij}$ [where $g_{00} = -f(r)$ for the class of geometries of eq. (2.1)] will be used as the relevant spatial metric [cf. eq. (1.4)] and the tilde denotes all geometric quantities defined therewith. As a result, in eq. (1.1), the differential operator becomes

$$\mathcal{K} = -\left(\tilde{\gamma}^{ij} \tilde{D}_i \tilde{D}_j + \tilde{E}\right) \tag{2.7}$$

and the eigenvalue is seen to be $\lambda = \omega^2$. In this approach, the operator (2.7) is built from the generalized covariant derivatives $\tilde{D}_j = \tilde{\nabla}_j + \tilde{\omega}_j$, with

$$\tilde{\omega}_i = \frac{(d-1)}{2}\, \partial_i \left(\ln \sqrt{|g_{00}|}\right) \tag{2.8}$$

obtained from the first-order terms in the conformally transformed Klein-Gordon equation. Incidentally, the sequence of rearrangements of eq. (2.6), leading to eq. (2.7) through the generalized covariant derivatives $\tilde{D}_j$, is equivalent to the Liouville transformations [17] of refs. [8, 9], thus solving the problem of transforming away the first-order derivatives. The one-form $\tilde{\omega}_i$ of eq. (2.8) can be reinterpreted as defining an additional vector-fiber structure [12] and plays the role of a connection. Moreover, eq. (2.8) shows it is an exact form, so that its associated curvature form $\tilde{\Omega}_{ij}$ vanishes—thus, the latter is absent from the heat-kernel expansion. Finally, the normal invariant or independent-term scalar in eq. (2.7),

$$\tilde{E} \equiv \tilde{E}(r) = -f(r)\,\left(m^2 + \xi R\right) - \tilde{\gamma}^{ij}\left(\partial_i \tilde{\omega}_j - \tilde{\omega}_l \tilde{\Gamma}^l_{ij} + \tilde{\omega}_i \tilde{\omega}_j\right)\,, \tag{2.9}$$

includes "extra terms" from the rearrangement of the generalized covariant derivative $\tilde{D}_j$.

The third step in the reduction procedure involves a number of subtleties that will be discussed in the next section.

3. Near-horizon heat-kernel approach

In this section we elaborate upon the two major technical building blocks of our *near-horizon spectral asymptotics*: the computation of the heat-kernel coefficients and the near-horizon approximation. Specifically, our hybrid framework involves a number of subtleties arising from the simultaneous use of the above ingredients.

Firstly, the computation of the heat-kernel or HaMiDeW coefficients involves the geometrically structured integrals [10, 12]

$$a_j \equiv a_j\left(\mathcal{K}\right) = \begin{cases} (4\pi)^{-d/2}\left[\int_{\mathcal{M}} d^dx\, \sqrt{\tilde{\gamma}}\, \Gamma_j^{(\mathcal{M})} + \int_{\partial\mathcal{M}} d^{d-1}x\, \sqrt{\tilde{h}}\, \Gamma_j^{(\partial\mathcal{M})}\right] & \text{if j is even}\,, \\ (4\pi)^{-(d-1)/2} \int_{\partial\mathcal{M}} d^{d-1}x\, \sqrt{\tilde{h}}\, \Gamma_j^{(\partial\mathcal{M})} & \text{if j is odd}\,, \end{cases} \tag{3.1}$$

which generalize eq. (1.4) and lead to extensions of Weyl's asymptotic formula (1.5) through eq. (1.3). The structural form and geometric content of eq. (3.1) are uniquely determined

by the optical metric (2.3) and the operator (2.7): the integrands $\Gamma_j^{(\mathcal{M})}$ and $\Gamma_j^{(\partial\mathcal{M})}$ are additively built from the primitive "HaMiDeW invariants," with appropriate combinatoric coefficients. In turn, the invariants are multiplicatively assembled, in all allowed geometric combinations at a given order, from the "geometrical building blocks" [10, 12]: (i) the Riemann tensor $\tilde{R}^{ij}{}_{kl}$; (ii) the normal invariant $\tilde{E}$; and (iii) the extrinsic curvature $\tilde{K}^a{}_b$ (with the curvature form $\tilde{\Omega}_{ij}$ being absent herein due to its identically vanishing value). In essence, from a given geometry, the "HaMiDeW operation" amounts to generating the set of coefficients (3.1), i.e., the formal operator $\mathfrak{H}\left[\mathcal{K}\right] = \left\{a_j(\mathcal{K})\right\}_{j\geq 0}$, whence the asymptotics of the spectral functions can be computed, e.g., $N_{\mathcal{K}}(\lambda)$ in eq. (1.3). This operation is displayed by the horizontal arrows in the diagram (3.2).

In addition, it should be noticed that, for the evaluation of eq. (3.1), the boundary is the event horizon—in this spatially reduced view, the horizon $\mathcal{H}$ represents the *spatial boundary* of the region classically accessible to an external observer. Thus, in a canonical approach with frequency-dependent spectral functions, the possible inclusion of terms arising from the spatial boundary $\mathcal{H}$ cannot be ignored.

Secondly, the other technical ingredient in our approach—which constitutes the third step of the reduction procedure of the previous section—amounts to isolating the dominant physics as $r \sim r_+$, i.e., performing an expansion in the near-horizon coordinate x of eq. (2.4) and abstracting the leading parts. This process, which will be represented by the symbol $\overset{(\mathcal{H})}{\sim}$, is displayed by the vertical arrows in the diagram (3.2). Moreover, the near-horizon hierarchical scheme serves a twofold purpose: (a) displaying the emergence of CQM as the leading physics of the modes that generate the field operator (2.5); (b) providing a consistent scaling of all the heat-kernel coefficients, which ultimately causes the holographic nature of the entropy. Therefore, this ingredient uncovers, *inter alia*, that CQM is responsible for the *universal holographic scaling* of spectral functions and the entropy. The resultant emergent behavior will be derived in the next section.

Having defined these two technical steps as formal operations, the question naturally arises as to the legitimacy of combining them into a unified approach and the order of their sequential application. More precisely, one may ask whether the diagram

$$
\begin{tikzcd}
\boxed{\begin{array}{c}\text{GLOBAL}\\ \text{GEOMETRY}\end{array}} \arrow[r, "\mathfrak{H}"] \arrow[d, "\overset{(\mathcal{H})}{\sim}"'] & \mathfrak{H}\left[\mathcal{K}\right] = \left\{a_j(\mathcal{K})\right\}_{j\geq 0} \arrow[d, "\overset{(\mathcal{H})}{\sim}"'] \\
\boxed{\begin{array}{c}\text{NEAR-HORIZON}\\ \text{GEOMETRY}\end{array}} \arrow[r, "\mathfrak{H}"] & \mathfrak{H}^{(\mathcal{H})}\left[\mathcal{K}\right] = \left\{a_j^{(\mathcal{H})}(\mathcal{K})\right\}_{j\geq 0}
\end{tikzcd}
\qquad (3.2)
$$

is indeed commutative. In essence, the legitimacy of the ordering consisting of the applica-

tion of $\mathfrak{H}$ followed by the near-horizon approximation $\stackrel{(\mathcal{H})}{\sim}$ is validated *a priori* by the large body of well-established results on the heat-kernel approach for compact manifolds [10]. Thus, the actual computations can be performed by introducing an infrared cutoff—indeed, this has been a standard procedure in the literature, as required by the usual statistical thermodynamic arguments (e.g., ref. [3]). When this is done, all the bulk integrals in eq. (3.1) are finite, except for possible divergences caused by the behavior of the metric upon crossing the horizon. However, the latter divergences (which amount to the "ultraviolet catastrophe" mentioned in section 1) can be regularized via a near-horizon brick-wall cutoff a, as further discussed and illustrated in the next section. Consequently, with the ultraviolet regulator a and an infrared thermodynamic cutoff, all the coefficients (3.1) are strictly finite.

As a result, the near-horizon expansion of the heat-kernel coefficients is justified as follows. Each coefficient involves integrals of the form $\int d\mu \, \Gamma$, which are hierarchically organized in powers of $x \equiv a$. The invariant integrands are built from the HaMiDeW invariants defined above, which are all finite at the horizon, even in the absence of a regulator—a result that is corroborated in the next section. By contrast, the measures $d\mu$ (both for the bulk and boundary contributions) are divergent and provide the leading scaling with respect to a; then, in the Landau big-O notation: $\Gamma = O(a^p)$, with $p \geq 0$, and $d\mu = O(a^q)$, with $q < 0$. The ensuing hierarchical process is used for the selection of the dominant order. Due to the finiteness of all expressions thus regularized, this process can be systematically applied to all the integrals and provides a result that is independent of the ordering in the diagram (3.2). The reversal of this ordering, with the near-horizon operation $\stackrel{(\mathcal{H})}{\sim}$ preceding the application of $\mathfrak{H}$, is explicitly used in the next section. Parenthetically, a more detailed evaluation of the global coefficients will be discussed in a forthcoming paper, along with explicit generalized expressions for the HaMiDeW invariants.

4. Near-horizon physics: building blocks & thermodynamics

In this section we display the emergence of CQM and the thermodynamics via the near-horizon heat-kernel formalism of the previous section.

The conformal behavior of the leading near-horizon physics is exposed from eq. (2.7) when $r \sim r_+$. We will only consider the *nonextremal* case, for which $f'_+ \equiv f'(r_+) \neq 0$ is related to the surface gravity $\kappa = f'_+/2$ and interpreted with the physics of CQM [8, 9], as shown below. Then, starting from $f(r) \stackrel{(\mathcal{H})}{\sim} f'_+ \, x$, the metric coefficients are $g_{00} \stackrel{(\mathcal{H})}{\sim} -f'_+ x$ and $g_{rr} \stackrel{(\mathcal{H})}{\sim} 1/\left(f'_+ x\right)$; thus, the radial and angular optical counterparts are $\tilde{\gamma}_{rr} \stackrel{(\mathcal{H})}{\sim} 1/\left(2\kappa x\right)^2$ and $\tilde{\gamma}_{ab} \stackrel{(\mathcal{H})}{\sim} \gamma_{ab}/\left(2\kappa x\right)$, while the Christoffel symbols are, e.g., $\tilde{\Gamma}^r_{rr} \stackrel{(\mathcal{H})}{\sim} -1/x$ and $\tilde{\Gamma}^r_{ab} \stackrel{(\mathcal{H})}{\sim} 2\kappa^2 \, \tilde{\gamma}_{ab} \, x$. In addition, the one-form (2.8) turns into

$$\tilde{\omega}_i \stackrel{(\mathcal{H})}{\sim} \frac{(d-1)}{4} \frac{1}{x} \, \delta^r_i \tag{4.1}$$

while the normal invariant (2.9), governed by the "extra terms," becomes

$$\tilde{E} \stackrel{(\mathcal{H})}{\sim} \frac{(d-1)^2}{4} \kappa^2 \; . \tag{4.2}$$

As a result, the rescaled leading form of the operator (2.7) becomes

$$-\frac{\mathcal{K}}{4\kappa^2} \stackrel{(\mathcal{H})}{\sim} x^2 \left[\partial_x - \frac{(d-5)}{4}\frac{1}{x}\right] \left[\partial_x + \frac{(d-1)}{4}\frac{1}{x}\right] + \frac{(d-1)^2}{16} = x^2 \partial_x^2 + x\partial_x \; , \tag{4.3}$$

which is manifestly *homogeneous of degree zero*. The conformal invariance of eq. (4.3) arises from the uniform degree of homogeneity of the kinetic and “extra terms,” while the surface gravity provides a scale for comparison with the eigenvalues of eq. (1.1), $\lambda = \omega^2$. Therefore, the *optical-metric version of near-horizon CQM* is given by eq. (1.1), with the operator (4.3); in addition, via Liouville transformations [17], this is equivalent to the other two forms of near-horizon CQM derived in ref. [9] (for example, the conformal coupling is $\Theta = \omega/f'_+$).

Most importantly, a concurrent near-horizon expansion extracts the leading behavior in x. This is unlike the orthodox procedure, in which the geometrical building blocks are evaluated from their global definitions over the manifold and its boundary. The first building block, the Riemann tensor, has a near-horizon *maximally symmetric* structure

$$\tilde{R}^{ij}{}_{kl} \stackrel{(\mathcal{H})}{\sim} -\kappa^2 \left(\delta^i_k \delta^j_l - \delta^i_l \delta^j_k\right) \; , \tag{4.4}$$

with its counterpart $\tilde{R}_{ijkl} \stackrel{(\mathcal{H})}{\sim} -\kappa^2 \left(g_{ik}g_{jl} - g_{il}g_{jk}\right)$. This symmetry is due to the single-parameter characterization of the near-horizon geometry by the *surface gravity* κ; i.e., κ sets uniquely the natural horizon scale of all physical quantities. The corresponding Ricci tensor $\tilde{R}^i{}_j \stackrel{(\mathcal{H})}{\sim} -(d-1)\,\kappa^2\,\delta^i_j$ and scalar curvature $\tilde{R} \stackrel{(\mathcal{H})}{\sim} -d\,(d-1)\,\kappa^2$ reflect the same symmetry; and as a result, contractions of multiple products of the three above Riemann-related quantities yield straightforward products and contractions of the corresponding Kronecker deltas and metric tensors. The second building block, the extrinsic curvature at the horizon,

$$\tilde{K}^a{}_b \stackrel{(\mathcal{H})}{=} \kappa\,\delta^a_b \; , \tag{4.5}$$

is also maximally symmetric due to the geometrically determining role of κ. Correspondingly, it generates the following contractions: $\tilde{K} \equiv \tilde{K}^a{}_a \stackrel{(\mathcal{H})}{=} (d-1)\,\kappa$, $\tilde{K}^a{}_b\,\tilde{K}^b{}_a \stackrel{(\mathcal{H})}{=} (d-1)\,\kappa^2$, and so on. The third building block, the normal invariant, displays the near-horizon leading behavior (4.2)—also parametrized with κ. Therefore, from Eqs. (4.2), (4.4), and (4.5), one infers the collective scaling property, with respect to the optical metric: (“Building Block”) $\stackrel{(\mathcal{H})}{\sim} \kappa^p \times O(x^0)$, where κ^p is solely determined by dimensional analysis from the scale κ. Thus, the HaMiDeW invariants and the invariant integrands inherit the same *near-horizon scaling* because they are assembled from the allowed contracted products of the building blocks (with the chosen placement of indices in $\tilde{R}^{ij}{}_{kl}$ and $\tilde{K}^a{}_b$); this results in the naive scaling

$$\hat{\Gamma}^{(\mathcal{M})}_j \equiv \frac{\Gamma^{(\mathcal{M})}_j}{\kappa^j} \stackrel{(\mathcal{H})}{\sim} O(x^0) \;\; , \hat{\Gamma}^{(\partial M)}_j \equiv \frac{\Gamma^{(\partial\mathcal{M})}_j}{\kappa^{j-1}} \stackrel{(\mathcal{H})}{\sim} O(x^0) \; . \tag{4.6}$$

As a corollary to eq. (4.6), the near-horizon behavior of the heat-kernel coefficients (3.1) is completely governed by the scaling of the integral measures: that of the bulk integral,

$$\begin{aligned}\tilde{\mathcal{V}}_d\left(\mathcal{M}\right) &\equiv \int_{\mathcal{M}} d^d x \sqrt{\tilde{\gamma}} \\ &\overset{(\mathcal{H})}{\sim} \frac{\mathcal{A}_{d-1}}{(2\kappa)^{(d+1)/2}} \int_a \frac{dx}{x^{(d+1)/2}} \overset{(\mathcal{H})}{\sim} \frac{1}{(d-1)\,\kappa^{(d+1)/2}}\, 2^{-(d-1)/2}\,\mathcal{A}_{d-1}\, a^{-(d-1)/2}\;, \end{aligned} \tag{4.7}$$

known as the "volume of optical space" [15], and that of the boundary integral,

$$\begin{aligned}\tilde{\mathcal{V}}_{d-1}\left(\partial\mathcal{M}\right) &\equiv \int_{\partial\mathcal{M}} d^{d-1} x \sqrt{\tilde{h}} \\ &\overset{(\mathcal{H})}{\sim} \frac{1}{\kappa^{(d-1)/2}}\, 2^{-(d-1)/2}\mathcal{A}_{d-1}\, a^{-(d-1)/2}\;, \end{aligned} \tag{4.8}$$

which are both regularized by means of the near-horizon radial-coordinate cutoff a. Because of the mandatory cutoff, the boundary $\partial\mathcal{M}$ is the horizon lifted an amount a in the radial coordinate r: this is the celebrated brick wall $\partial\mathcal{M} = \mathcal{H}_a$. Furthermore, the regularized integrals in Eqs. (4.7) and (4.8) can be rendered into the invariant forms

$$\tilde{\mathcal{V}}_{D-1}\left(\mathcal{M}\right) \overset{(\mathcal{H})}{\sim} \frac{4}{(D-2)}\,[h_D]^{-(D-2)}\,\frac{\mathcal{A}_{D-2}}{4}\,\kappa^{-(D-1)} \tag{4.9}$$

and

$$\tilde{\mathcal{V}}_{D-2}\left(\partial\mathcal{M}\right) \overset{(\mathcal{H})}{\sim} 4\,[h_D]^{-(D-2)}\,\frac{\mathcal{A}_{D-2}}{4}\,\kappa^{-(D-2)}\;, \tag{4.10}$$

via the geometrical radial distance $h_D \overset{(\mathcal{H})}{\sim} 2\sqrt{a/f'_+} = \sqrt{2a/\kappa}$ and the horizon area $\mathcal{A}_{D-2}$ in D spacetime dimensions. In addition, for the remainder of the paper, we will use the spacetime dimensionality $D = d+1$. Then, Eqs. (3.1), (4.9), and (4.10) imply that the dimensionless coefficients $\hat{a}_j \equiv a_j\,\kappa^{D-j-1}$ are given by

$$\hat{a}_j = \frac{2^{3-D}}{\pi^{(D-1)/2}}\,\hat{I}_D^{(j)}\,[h_D]^{-(D-2)}\,\frac{\mathcal{A}_{D-2}}{4}\;, \tag{4.11}$$

where

$$\hat{I}_D^{(j)} = \begin{cases} \frac{1}{(D-2)}\,\hat{\Gamma}_j^{(\mathcal{M})} + \hat{\Gamma}_j^{(\partial\mathcal{M})} & \text{if j is even}\;, \\ \sqrt{4\pi}\,\hat{\Gamma}_j^{(\partial\mathcal{M})} & \text{if j is odd}\;. \end{cases} \tag{4.12}$$

The main physical quantity of interest in this paper is the entropy, which we derive through the expansion (1.3) of the spectral function, within the spectral rule [8, 9]

$$S = -\int_0^\infty d\omega\,\ln(1-e^{-\beta\omega})\,\left[\left(\omega\frac{d}{d\omega}+2\right)\frac{dN(\omega)}{d\omega}\right]\;. \tag{4.13}$$

Then, evaluating the dimensionless integrals in terms of the Riemann zeta function $\zeta_R(z)$ and enforcing the inverse Hawking temperature $\beta \equiv \beta_H = 2\pi/\kappa$, the leading near-horizon part of the entropy takes the form

$$S \overset{(\mathcal{H})}{\sim} \sum_{j=0}^{\infty} \frac{(D-j-1)\,(D-j)}{(2\pi)^{D-j-1}}\,\frac{\zeta_R(D-j)\,\Gamma(D-j-1)}{\Gamma\big((D-j+1)/2\big)}\,\hat{a}_j\;. \tag{4.14}$$

The remarkable conclusion of this analysis is that *all the terms in the HaMiDeW near-horizon expansion of the entropy, $S^{(j)}$, are structurally similar, being governed by CQM and generating the holographic property $S \propto \mathcal{A}_{D-2}$*. Specifically,

$$S^{(j)} = c_D^{(j)} \, \hat{I}_D^{(j)} \, [h_D]^{-(D-2)} \, \frac{\mathcal{A}_{D-2}}{4} \, , \tag{4.15}$$

which involves the dimensionless prefactors given by $\hat{I}_D^{(j)}$ and

$$c_D^{(j)} = \frac{2^{4-D}}{\pi^{3D/2-1-j}} \, \Gamma\left[\frac{(D-j)}{2} + 1 \right] \zeta_R \left(D - j \right) \, . \tag{4.16}$$

Incidentally, even though the HaMiDeW numerical coefficients are of order unity, they are also functions of the dimensionality D; thus, they could be zero for particular values of D. For example, the lowest-order dimensionless invariant integrands of the heat kernel coefficients—evaluated with the usual *Dirichlet boundary condition at the brick wall*—are: $\hat{\Gamma}_0^{(\mathcal{M})} \overset{(\mathcal{H})}{\sim} 1$, $\hat{\Gamma}_1^{(\partial\mathcal{M})} \overset{(\mathcal{H})}{\sim} -1/4$, $\hat{\Gamma}_2^{(\mathcal{M})} \overset{(\mathcal{H})}{\sim} (D-2)(D-4)/12$, $\hat{\Gamma}_2^{(\partial\mathcal{M})} \overset{(\mathcal{H})}{\sim} (D-2)/3$, and so on. Equation (4.13) suggests that the terms $j \leq D-1$ are relevant, while the higher orders require further analysis. The ensuing pattern of cancellations and the significance of higher orders for different dimensionalities will be systematically studied elsewhere.

5. Context and conclusions

A more detailed comparison of our near-horizon heat-kernel framework with the existing literature is in order. An important point of contact is established by exploring the temperature expansion for miscellaneous thermodynamic functions. This can be developed in the *ab initio* canonical framework from the series (1.3) with the spectral counting function $N(\omega)$, via the counterpart of eq. (4.13) for a generic thermodynamic function, i.e.,

$$\mathcal{T} = - \int_0^\infty d\omega \, \ln(1 - e^{-\beta\omega}) \, \hat{\mathcal{T}}_\omega(\beta) \left[\frac{dN(\omega)}{d\omega} \right] \, , \tag{5.1}$$

where the operator $\hat{\mathcal{T}}_\omega(\beta)$ (differential with respect to ω and possibly β-dependent) is function-specific. Most importantly, for the free energy $\mathcal{T} = F$, $\hat{\mathcal{T}}_\omega = -1/\beta$; and for the entropy $\mathcal{T} = S$, $\hat{\mathcal{T}}_\omega = 2 + \omega \, d/d\omega$. Substituting eq. (1.3) in eq. (5.1), one may write $\mathcal{T} = \sum_{j=0}^\infty \mathcal{T}^{(j)}$ [with j being the order in the heat-kernel series (1.3)], which yields a temperature expansion for all thermodynamic functions. The scaling of these functions, order by order, follows by using the same approach as that leading to the scaling of eq. (4.13)—the only caveat is that, for comparison purposes, here we will not assume that the temperature is the Hawking temperature yet. From eq. (5.1), $\mathcal{T}^{(j)} \propto \beta^{2r_j(d)+\deg_\beta(\hat{\mathcal{T}}_\omega)}$, with $\deg_\beta(\hat{\mathcal{T}}_\omega)$ being the degree of homogeneity of the operator $\hat{\mathcal{T}}_\omega$ with respect to β. For example, $S^{(j)} \propto \beta^{2r_j(d)} = T^{D-1-j}$ for the entropy [i.e., $S^{(j)}$ is eq. (4.15) times $(\beta\kappa/2\pi)^{2r_j(d)}$] while $F^{(j)} \propto \beta^{2r_j(d)-1} = T^{D-j}$ for the free energy; more precisely.

$$-F^{(j)} \overset{(\mathcal{H})}{\sim} \frac{1}{(D-j)\,\beta} \, S^{(j)} \overset{(\mathcal{H})}{\sim} \frac{c_D^{(j)}}{(D-j)} \, \hat{I}_D^{(j)} \, [h_D]^{-(D-2)} \, \frac{\mathcal{A}_{D-2}}{4} \left(\frac{\kappa}{2\pi} \right)^{j-D+1} \beta^{j-D} \, . \tag{5.2}$$

In particular, the leading order, $j = 0$: (i) is equivalent (by definition) to the thermodynamics based on Weyl's formula (1.5); (ii) is seen to reproduce the standard brick-wall results of refs. [3] and [4]. Moreover, this a term that is typically singled out in refs. [15, 16, 18, 19] [for example, eq. (3.26) in ref. [15], as can be verified from our Eqs. (4.9) and (5.2)].

Our work shows that usage of the high-temperature limit or equivalent for black hole thermodynamics may be unwarranted because the Hawking temperature is of order unity in surface-gravity units. Once the comparison above is made, one may ask the extent to which these results modify the standard folklore for the thermodynamics in the presence of horizons. The key lesson from this computation is the realization that, conceptually, the scaling of all orders of the HamiDeW expansion is identical when the Hawking temperature is enforced; thus, by the same token, this implies in the end that the holographic scaling of the entropy remains intact. In a sense, the holographic outcome is enhanced—in the language we have often used in this paper. In the brick-wall approach, this specifically yields a correction for the expression of the brick-wall elevation, with the main conclusions of refs. [3] and [4] remaining intact. Possible additional implications of these differences are discussed next.

In conclusion, we have developed a near-horizon heat-kernel framework that comprehensively addresses the derivation of black hole thermodynamics from conformal quantum mechanics. The guiding strategy has been the inclusion of all contributions to the density of modes within a brick-wall type model. In essence, this hybrid framework combines the conformal properties of the horizon with the insight afforded by geometric spectral asymptotics; such format transcends the traditional heat-kernel hierarchy and the standard brick-wall model. As an outcome of these extensions, we have found that all orders of the heat-kernel expansion and associated spectral functions, e.g., the density of modes, exhibit the same scaling behavior with respect to the near-horizon expansion. At the level of the entropy, the ensuing conformal enhancement further stresses its holographic nature. In addition, as these findings entail possible modifications of the usual brick-wall type calculations, they should be included in any discussions of the "species problem" [3, 2, 4] and the renormalization of Newton's gravitational constant [15, 16, 18, 19]. Moreover, the robustness of holographic scaling and the crucial role of scaling and conformal properties for black hole thermodynamics call for a deeper interpretation, possibly related to conformal field theory [20, 21]. These unresolved issues, as well as an extension of scaling properties via path integrals, including a detailed comparison with refs. [14, 15, 16], will be addressed in a forthcoming paper.

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